On Complexity Reduction of the LP Bound Computation and Related Problems

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Introduction

- Random variables $\mathcal{V} = \{A, B, ...\}$, $|\mathcal{V}| = n$
- Entropy $h : \mathcal{P}(\mathcal{V}) \mapsto \mathbb{R}$

$$\mathbf{h} = [h(X_{\mathcal{A}}) : \mathcal{A} \subseteq \mathcal{N} \setminus \emptyset]^{\mathsf{T}} \in \mathbb{R}^{2^n - 1}$$

- Elemental inequalities [Yeung 1997]

$$h(A|\mathcal{V} \setminus \{A\}) \geq 0, A \in \mathcal{V}$$
$$I(A; B|\mathcal{C}) \geq 0, A \neq B \neq \emptyset, \mathcal{C} \subseteq \mathcal{V} \setminus \{A, B\}$$
$$m = n + \binom{n}{2} 2^{n-2}$$
$$\Gamma = \{\mathbf{h} : \mathbf{G}\mathbf{h} \geq 0\}$$

- $\mathbf{G}$ is an $m \times (2^n - 1)$ matrix and $\mathbf{h}$ is a $2^n - 1$ column vector.
- Region Γ is the intersection of half-spaces in $\mathbb{R}^{2^n - 1}$

The region Γ is of fundamental importance

- Weighted sum-rate LP bound

$$\text{maximize } \mathbf{w}^{\mathsf{T}}\mathbf{h} \text{ subject to } \begin{cases} \mathbf{G}\mathbf{h} & \geq 0 \\ \mathbf{c}_1^{\mathsf{T}}\mathbf{h} & = 0 \\ \mathbf{C}_2\mathbf{h} & = 0 \\ \mathbf{C}_3\mathbf{h} & = 0 \\ \mathbf{J}\mathbf{h} & \leq \mathbf{c}_4 \end{cases}$$

- Proving basic information inequalities: redundancy check
- Computer program: ITIP [Yeung and Yan]

$$\text{minimize } \mathbf{b}^{\mathsf{T}}\mathbf{h}, \text{ subject to } \begin{cases} \mathbf{G}\mathbf{h} & \geq 0 \\ \mathbf{C}\mathbf{h} & = 0 \end{cases}$$

Computing is a formidable task

- Dimensions $= 2^n - 1$
- Constraints $> m = n + \binom{n}{2} 2^{n-2}$
- Exhausts computational and memory resources
- Simplex Algorithm: Exponential worst case complexity in size
- Means, doubly exponential in n

Contributions

- Characterized the feasible regions with significantly less inequalities

$$\left\{ \mathbf{h} : \begin{array}{rl} \mathbf{G}\mathbf{h} & \geq 0, \\ \mathbf{C}_2\mathbf{h} & = 0, \\ \mathbf{C}_3\mathbf{h} & = 0 \end{array} \right\} = \Gamma \cap \mathcal{C}_2 \cap \mathcal{C}_3 \Longleftrightarrow \Upsilon \cap \Omega = \left\{ \mathbf{h} : \begin{array}{rl} \mathbf{M}\mathbf{h} & \geq 0, \\ \mathbf{K}\mathbf{h} & = 0, \\ \mathbf{L}\mathbf{h} & = 0 \end{array} \right\}$$

- Algorithms to *directly* generate the systems of reduced inequalities and equalities

- Upper bounds on the size of the systems of reduced inequalities and equalities

Outline

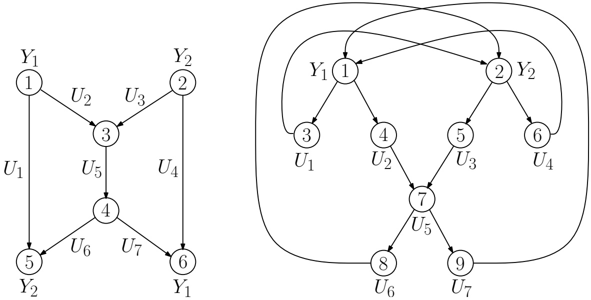
1. Irreducible sets
2. Reduction in Elemental Inequalities
3. Algorithms
4. Upper Bounds on the size
5. Butterfly Network

Functional Dependence Graphs

Definition (Functional Dependence Graph)

Let $\mathcal{V} = \{A, B, \dots\}$ be a set of random variables with entropy function h. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a *functional dependence graph* for $\mathcal{V}$ if and only if for all $B \in \mathcal{V}$

$$h\left(B \mid \{A : (A, B) \in \mathcal{E}\}\right) = 0.$$



Irreducible sets

- Notion of functional dependence on Functional Dependence Graph

- $\mathcal{A}$ determines $\mathcal{B}$ in FDG $\mathcal{G}(\mathcal{V},\mathcal{E})$: $\mathcal{A} \to \mathcal{B}$

$$\mathcal{A} \to \mathcal{B} \implies h(\mathcal{B} \mid \mathcal{A}) = 0$$

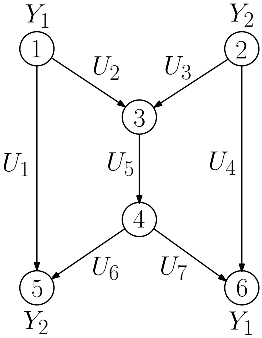
- $\phi(\mathcal{A})$ is the largest set of nodes with $\mathcal{A} \to \phi(\mathcal{A})$.

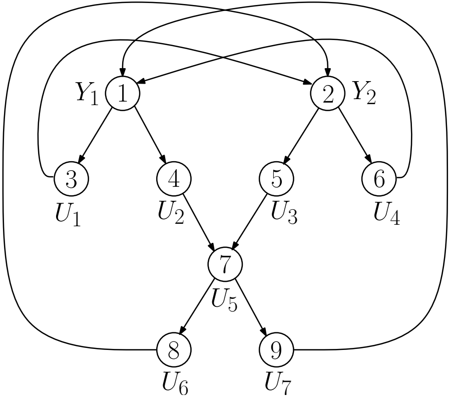
- Irreducible set: A set of nodes $\mathcal{B}$ in a functional dependence graph is *irreducible* if there is no $\mathcal{A} \subset \mathcal{B}$ with $\mathcal{A} \to \mathcal{B}$.

- An irreducible set $\mathcal{B}$ is maximal if $\phi(\mathcal{B}) = \mathcal{V}$.

S. Thakor, A. Grant and T. Chan.
Network coding capacity: A functional dependence bound.
in *International Symposium on Information Theory*, pp.263–267, Jul 2009.

Example





$$Y_1 \longrightarrow \{U_1, U_2\} \qquad \Longrightarrow h(U_1U_2Y_1) = h(Y_1)$$

$$\{Y_1, Y_2\} \longrightarrow \mathcal{V} \qquad \Longrightarrow h(\mathcal{V}) = h(Y_1Y_2)$$

Reduction in Elemental Inequalities

- Notations: the set of all irreducible sets $\mathcal{K}$ and the set of all maximal irreducible sets $\boldsymbol{\mathcal{M}} \subset \mathcal{K}$.

Lemma

Let $A \in \mathcal{V}$. *If there exists* $\mathcal{B} \in \boldsymbol{\mathcal{M}}$ *such that* $\mathcal{B} \subseteq \mathcal{V} \setminus \{A\}, \mathcal{B} \neq \mathcal{S}$ *then* $h(A|\mathcal{V} \setminus \{A\}) \geq 0$ *can be replaced by*

$$h(\mathcal{S}) - h(\mathcal{B}) = 0$$

where $\mathcal{B} \neq \mathcal{S} \in \boldsymbol{\mathcal{M}}$. *Otherwise, if no such* $\mathcal{B}$ *exists then the elemental non-decreasing inequality can be replaced by*

$$h(\mathcal{S}) - h(\mathcal{B}) \geq 0$$

where $\mathcal{B} \subseteq \mathcal{V} \setminus \{A\}, \mathcal{V} \setminus \{A\} \subseteq \phi(\mathcal{B}), \mathcal{B} \in \mathcal{K} \setminus \boldsymbol{\mathcal{M}}$ *and* $\mathcal{S} \in \boldsymbol{\mathcal{M}}$.

- Modification of the inequalities, no reduction.

Lemma

For $A, B \in \mathcal{V}$ *and* $\mathcal{C} \subseteq \mathcal{V} \setminus \{A, B\}$, *if* $\mathcal{C}$ *is not irreducible then*

$$I(A;B|\mathcal{C}) = I(A;B|\mathcal{B}) \tag{i}$$

where $\mathcal{B} \subset \mathcal{C} : \mathcal{C} \subseteq \phi(\mathcal{B}), \mathcal{B} \in \mathcal{K}$.

- Proof

$$\begin{aligned} I(A;B|\mathcal{C}) &= h(A\mathcal{C}) + h(B\mathcal{C}) - h(AB\mathcal{C}) - h(\mathcal{C}) \\ &= h(A\mathcal{B}) + h(B\mathcal{B}) - h(AB\mathcal{B}) - h(\mathcal{B}) \\ &= I(A;B|\mathcal{B}) \end{aligned}$$

- Significant reduction in the number of submodular inequalities

Lemma

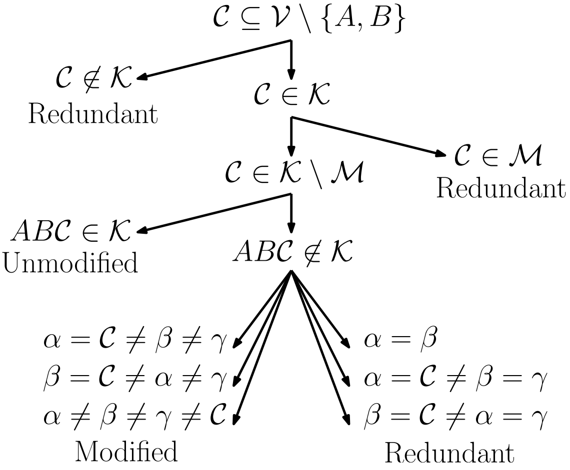
For $A, B \in V$ *and* $\mathcal{C} \subseteq \mathcal{V} \setminus \{A, B\}$, *let* $\mathcal{C}$ *be a maximal irreducible set then the elemental submodular inequality* $I(A; B|\mathcal{C}) \geq 0$ *is redundant. That is*

$$I(A; B|\mathcal{C}) \geq 0, \mathcal{C} \in \boldsymbol{\mathcal{M}} \tag{ii}$$

are redundant inequalities.

- Proof

$$\begin{aligned} I(A; B|\mathcal{C}) &= h(A\mathcal{C}) - h(B\mathcal{C}) - h(AB\mathcal{C}) - h(\mathcal{C}) \\ &= h(\mathcal{C}) - h(\mathcal{C}) - h(\mathcal{C}) - h(\mathcal{C}) \\ &= 0 \end{aligned}$$



Cases and their effect on elemental submodular inequalities $I(A; B|\mathcal{C}) \geq 0$.

Corollary

For $A, B \in V$ *and* $\mathcal{C} \subseteq \mathcal{V} \setminus \{A, B\}$*, let* $AB\mathcal{C}$ *be irreducible. If* $AB\mathcal{C} \in \boldsymbol{\mathcal{M}}$ *then the elemental submodular inequalities* $I(A; B|\mathcal{C}) \geq 0$ *can be replaced by*

$$h(A\mathcal{C}) + h(B\mathcal{C}) - h(\mathcal{S}) - h(\mathcal{C}) \geq 0$$

where, $\mathcal{S} \in \boldsymbol{\mathcal{M}}$ *is the set of all source pseudo-variables.*

- Now, consider the case: $\mathcal{C}$ is irreducible but $AB\mathcal{C}$ is not irreducible. To study different cases within this case, we define the following set.

$$\boldsymbol{\mathcal{X}}(\mathcal{A}) \triangleq \{\alpha \subseteq \mathcal{A} : \alpha \in \boldsymbol{\mathcal{K}}, \mathcal{A} \subseteq \phi(\alpha)\}$$

- Note that the pseudo-entropy of the set of pseudo-variables $\mathcal{A}$ can be replaced by the pseudo-entropy of any set $\alpha \in \boldsymbol{\mathcal{X}}(\mathcal{A})$.

Lemma

For reducible set $AB\mathcal{C}$ and irreducible set $\mathcal{C}$, if there exists $\alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C})$ such that $\alpha = \beta$ then the elemental inequality $I(A; B|\mathcal{C}) \geq 0$ is redundant. That is,

$$I(A; B|\mathcal{C}) \geq 0 : \begin{array}{l} AB\mathcal{C} \notin \boldsymbol{\mathcal{K}}, \mathcal{C} \in \boldsymbol{\mathcal{K}}, \\ \alpha = \beta, \\ \alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C}) \end{array}$$

are redundant inequalities.

Lemma

For reducible sets ABC *and irreducible set* $\mathcal{C}$*, if there exists* $\alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C})$ *and* $\gamma \in \boldsymbol{\mathcal{X}}(AB\mathcal{C})$ *such that* $\alpha = \mathcal{C} \neq \beta = \gamma$ *or* $\beta = \mathcal{C} \neq \alpha = \gamma$ *then the elemental inequality* $I(A;B|\mathcal{C}) \geq 0$ *is redundant. That is,*

$$I(A;B|\mathcal{C}) \geq 0 : \begin{array}{l} AB\mathcal{C} \notin \boldsymbol{\mathcal{K}}, \mathcal{C} \in \boldsymbol{\mathcal{K}}, \\ \alpha = \mathcal{C} \neq \beta = \gamma, \\ \alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C}), \gamma \in \boldsymbol{\mathcal{X}}(AB\mathcal{C}) \end{array}$$

$$I(A;B|\mathcal{C}) \geq 0 : \begin{array}{l} AB\mathcal{C} \notin \boldsymbol{\mathcal{K}}, \mathcal{C} \in \boldsymbol{\mathcal{K}}, \\ \beta = \mathcal{C} \neq \alpha = \gamma \\ \alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C}), \gamma \in \boldsymbol{\mathcal{X}}(AB\mathcal{C}) \end{array}$$

are redundant inequalities since they can be written as trivial equalities.

Observe

- By proving inequalities to be redundant, we have reduced number of required inequalities.
- By modifying inequalities, we have reduced the dimension (the number of variables) of the required inequalities.

Algorithm

ReducedLPIneqCS($\mathcal{V}, \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}, \mathcal{S} \in \boldsymbol{\mathcal{M}}$)

Require: $\mathcal{V}, \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}, \mathcal{S} \in \boldsymbol{\mathcal{M}}$

$\mathcal{I} = \emptyset$
for all $A \in \mathcal{V}$ **do**
  if $\exists \mathcal{B} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}} : \mathcal{B} \subseteq \mathcal{V} \setminus \{A\} \subseteq \phi(\mathcal{B})$ **then**
    $\mathcal{I} \leftarrow \mathcal{I} \cup \{h(\mathcal{S}) - h(\mathcal{B}) \geq 0\}$
  end if
end for
for all $\{A, B\} \subset \mathcal{V}, A \neq B \neq \emptyset$ **do**
  for all $\mathcal{C} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}} : A, B \notin \mathcal{C}$ **do**
    if $\{A, B, \mathcal{C}\} \in \boldsymbol{\mathcal{M}}$ **then**
      $\mathcal{I} \leftarrow \mathcal{I} \cup \{h(A\mathcal{C}) + h(B\mathcal{C}) - h(\mathcal{S}) - h(\mathcal{C}) \geq 0\}$
    else if $\{A, B, \mathcal{C}\} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}}$ **then**
      $\mathcal{I} \leftarrow \mathcal{I} \cup \{h(A\mathcal{C}) + h(B\mathcal{C}) - h(AB\mathcal{C}) - h(\mathcal{C}) \geq 0\}$
    else if

$$\nexists \alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C}), \gamma \in \boldsymbol{\mathcal{X}}(AB\mathcal{C}) : \begin{array}{l} \alpha = \beta = \gamma = \mathcal{C}, \\ \alpha = \mathcal{C} \neq \beta = \gamma, \\ \beta = \mathcal{C} \neq \alpha = \gamma \end{array}$$

    then
      Pick any $\alpha \in \boldsymbol{\mathcal{X}}(A\mathcal{C}), \beta \in \boldsymbol{\mathcal{X}}(B\mathcal{C}), \gamma \in \boldsymbol{\mathcal{X}}(AB\mathcal{C})$ and
      $\mathcal{I} \leftarrow \mathcal{I} \cup \{h(\alpha) + h(\beta) - h(\gamma) - h(\mathcal{C}) \geq 0\}$
    end if
  end for
end for
Output $\mathcal{I}$

Algorithm

ReducedLPEqCS($\mathcal{V}, \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}}, \mathcal{S} \in \boldsymbol{\mathcal{M}}$)

Require: $\mathcal{V}, \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{M}} \subset \boldsymbol{\mathcal{K}}$

$\mathcal{J} = \emptyset$
for all $\mathcal{B} \in \boldsymbol{\mathcal{M}} \setminus \mathcal{S}$ **do**
  $\mathcal{J} \leftarrow \mathcal{J} \cup \{h(\mathcal{S}) - h(\mathcal{B}) = 0\}$
end for
for all $\mathcal{A}, \mathcal{B} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}} : \mathcal{A} \neq \mathcal{B}, \mathcal{A} \cup \mathcal{B} \notin \boldsymbol{\mathcal{K}}$ **do**
  if $\mathcal{A} \cup \mathcal{B} \subseteq \phi(\mathcal{A})$ and $\mathcal{A} \cup \mathcal{B} \subseteq \phi(\mathcal{B})$ **then**
    $\mathcal{J} \leftarrow \mathcal{J} \cup \{h(\mathcal{A}) - h(\mathcal{B}) = 0\}$
  end if
end for
$\mathcal{J} \leftarrow \textbf{ReducedRowEchelon}(\mathcal{J})$
for all $\mathcal{C} \subseteq \mathcal{V} : \mathcal{C} \notin \boldsymbol{\mathcal{K}}$ **do**
  Find a set $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{C} \subseteq \phi(\mathcal{B}), \mathcal{B} \in \boldsymbol{\mathcal{K}}$
  $\mathcal{J} \leftarrow \mathcal{J} \cup \{h(\mathcal{C}) - h(\mathcal{B}) = 0\}$
end for
Output $\mathcal{J}$

The new systems of inequalities and equalities

$$\mathbf{MKh} \geq 0 \triangleq \left\{ \begin{array}{rl} h(\mathcal{S}) - h(\mathcal{B}) \geq 0 & : \mathcal{S} \in \boldsymbol{\mathcal{M}}, \mathcal{B} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}}, \\ & \ \ \mathcal{B} \subseteq \mathcal{V} \setminus \{A\} \subseteq \phi(\mathcal{B}_j), A \in \mathcal{V} \\ I(A; B|\mathcal{C}) \geq 0 & : ABC \in \boldsymbol{\mathcal{K}} \\ h(\beta) - h(\gamma) \geq 0 & : \alpha = \mathcal{C} \neq \beta \neq \gamma \\ h(\alpha) - h(\gamma) \geq 0 & : \beta = \mathcal{C} \neq \alpha = \gamma \\ h(\alpha) + h(\beta) - h(\gamma) - h(\mathcal{C}) \geq 0 & : \alpha \neq \beta \neq \gamma \neq \mathcal{C} \end{array} \right\}$$

$$\mathbf{Kh} = 0 \triangleq \left\{ \begin{array}{rl} h(\mathcal{S}) - h(\mathcal{B}) = 0 & : \mathcal{B} \in \boldsymbol{\mathcal{M}} \setminus \mathcal{S} \\ h(\mathcal{B}) - h(\mathcal{C}) = 0 & : \mathcal{B}, \mathcal{C} \in \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}}, \mathcal{B} \cup \mathcal{C} \notin \boldsymbol{\mathcal{K}}, \\ & \ \ \mathcal{B} \cup \mathcal{C} \subseteq \phi(\mathcal{B}), \mathcal{B} \cup \mathcal{C} \subseteq \phi(\mathcal{C}) \end{array} \right\}$$

$$\mathbf{Lh} = 0 \triangleq \left\{ h(\mathcal{C}) - h(\mathcal{B}) = 0 : \ \mathcal{C} \notin \boldsymbol{\mathcal{K}}, \mathcal{B} \in \boldsymbol{\mathcal{K}}, \mathcal{B} \subset \mathcal{C} \subseteq \phi(\mathcal{B}) \ \right\}$$

- Define

$$\Upsilon \triangleq \{\mathbf{h} : \mathbf{M}\mathbf{h} \geq 0\}$$
$$\Omega \triangleq \{\mathbf{h} : \mathbf{K}\mathbf{h} = 0, \mathbf{L}\mathbf{h} = 0\}$$

- The intersection

$$\Upsilon \cap \Omega = \{\mathbf{h} : \mathbf{M}\mathbf{h} \geq 0, \mathbf{K}\mathbf{h} = 0, \mathbf{L}\mathbf{h} = 0\}$$

Theorem

The region $\Gamma \cap \mathcal{C}_2 \cap \mathcal{C}_3$ *and the region* $\Upsilon \cap \Omega$ *are the same.*

Upper bounds on the size

Lemma

Given the set of all maximal irreducible sets $\mathcal{M}$ *and the set of all irreducible sets* $\mathcal{K} \supset \mathcal{M}$*, the number of inequalities describing the region* Υ *is upper bounded by*

$$n + \binom{n}{2}|\mathcal{K} \setminus \mathcal{M}|$$

and the number of dimensions describing the region Υ *is upper bounded by*

$$|\mathcal{K} \setminus \mathcal{M}|.$$

The problems reformulated

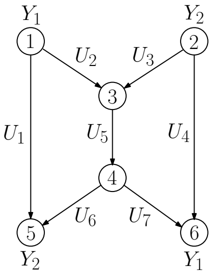
- Weighted sum-rate LP bound

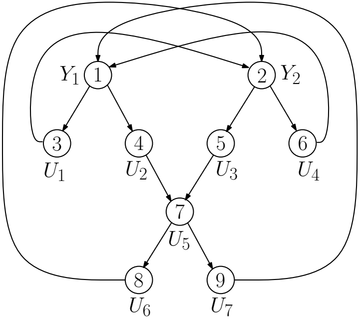
$$\text{maximize } \mathbf{w}^\mathsf{T}\mathbf{h} \text{ subject to } \begin{cases} \mathbf{M}\mathbf{h} \geq 0 \\ \mathbf{c}_1^\mathsf{T}\mathbf{h} = 0 \\ \mathbf{J}\mathbf{h} \leq \mathbf{c}_4 \end{cases}$$

- Redundancy check (ITIP)

$$\text{minimize } \mathbf{b}^\mathsf{T}\mathbf{h} \text{ subject to } \mathbf{M}\mathbf{h} \geq 0$$

Butterfly Network





$$
\begin{aligned}
\boldsymbol{\mathcal{M}} = \{&\{1,2\},\{1,5\},\{1,7\},\{1,8\},\{2,4\},\{2,7\},\{2,9\},\{3,4,5\},\\
&\{3,4,8\},\{3,7\},\{3,8,9\},\{4,5,6\},\{5,6,9\},\{6,7\},\{6,8,9\}\}\\
\boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}} = \{&\emptyset,\{1\},\{2\},\{13\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{1,6\},\{1,9\},\\
&\{2,3\},\{2,8\},\{3,4\},\{3,5\},\{3,6\},\{3,8\},\{3,9\},\{4,5\},\{4,6\},\\
&\{4,7\},\{4,8\},\{4,9\},\{5,6\},\{5,7\},\{5,8\},\{5,9\},\{6,8\},\{6,9\},\\
&\{8,9\},\{3,4,6\},\{3,4,9\},\{3,5,6\},\{3,5,9\},\{4,6,8\},\{4,8,9\},\\
&\{5,6,8\},\{5,8,9\}\}
\end{aligned}
$$

Butterfly Network

The Original LP

- Dimensions: $2^9 - 1 = 511$
- Constraints: $14 + 9 + \binom{9}{2}2^7 = 4631$

The reduced LP

- Dimensions: $|\boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}}| = 39$
- The upper bound on the number of inequalities: $9 + \binom{9}{2}39 = 1413$
- The number of inequalities generated by the algorithm: 844
- The total number of inequalities for the LP bound computation: 852

Butterfly Network

- The size of the constraints matrix is reduced from 4631×511 to 852×39.
- 92% reduction in the number of dimensions,
- 81% reduction in the number of constraints and
- 98% reduction in the size of the problem.

Q & A