Delay-Optimal Burst Erasure Codes for Parallel Links

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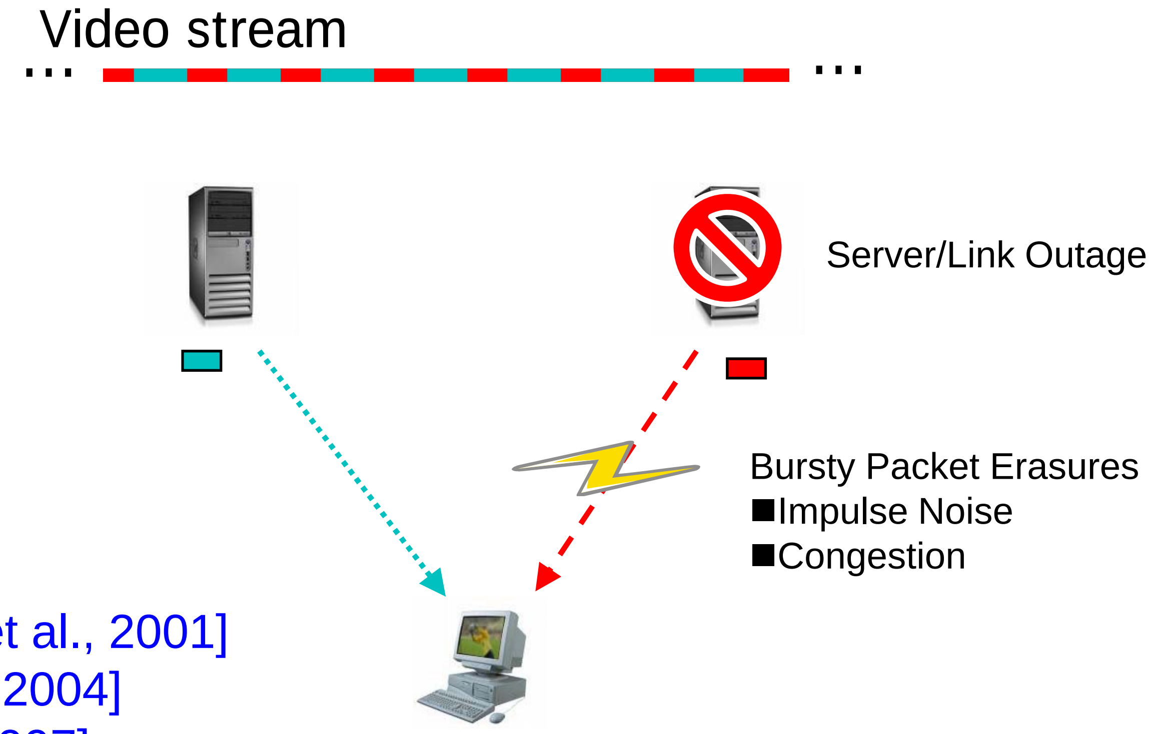
Outline

- Motivations
  - e.g. video streaming with path diversity
- Related work
- Problem formulation
- Main result – rate-delay tradeoff
  - Achievability – practical code construction
  - Converse – entropy argument
- Extension to single-source multicast

Multiple-Sender Video Streaming



[Apostolopulous et al., 2001]
[Nguyen, Zakhor, 2004]
[Frossard et al., 2007]
……

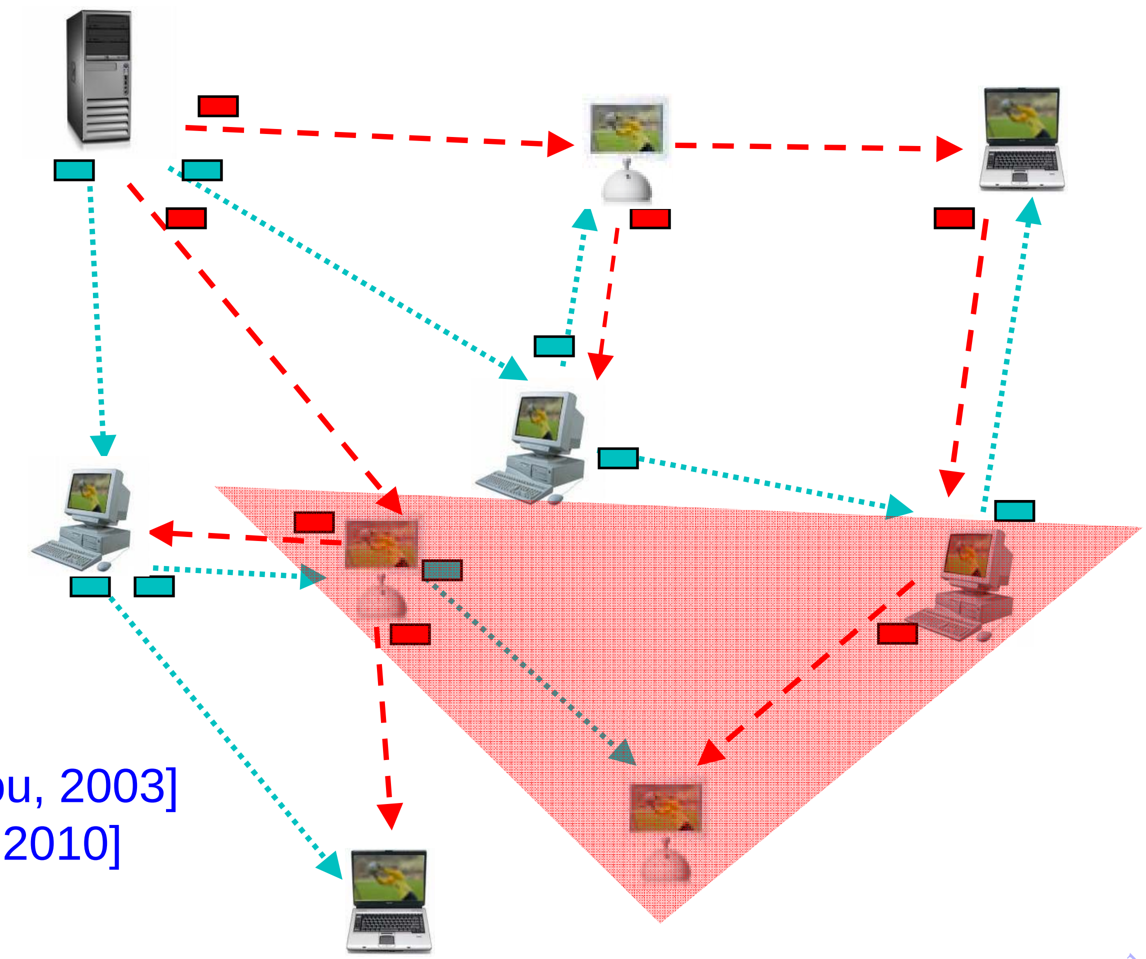
P2P Live Multicast with Complementary Trees

Video stream

... ...

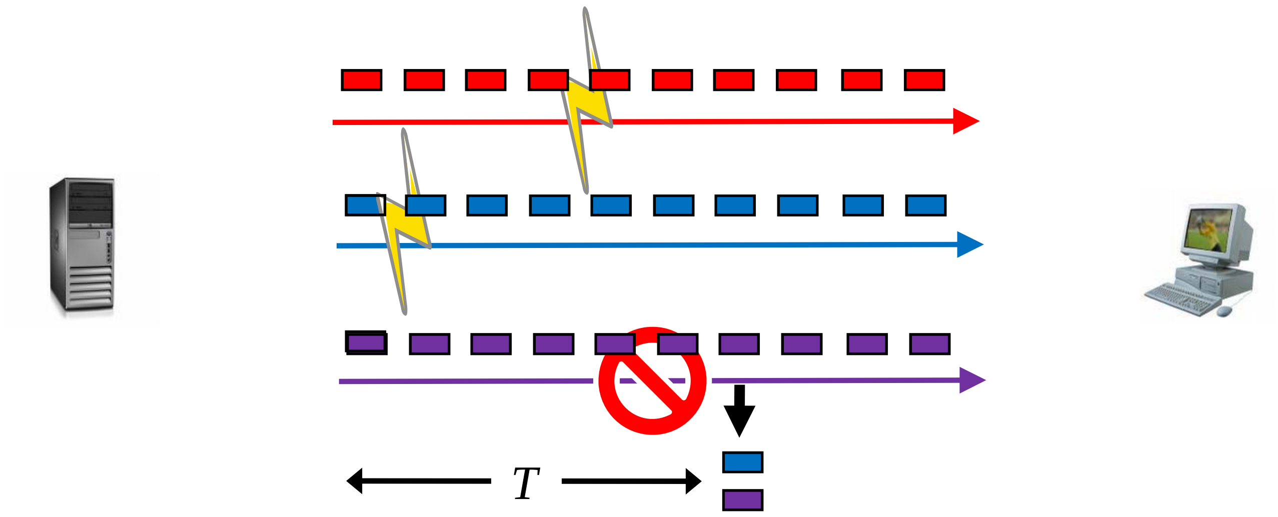
Tree 1

Tree 2



[Padmanabhan, Wang, Chou, 2003]
[Setton, Noh, Girod, 2005 - 2010]
……

Erasure Coding for Packets Streamed over Parallel Links

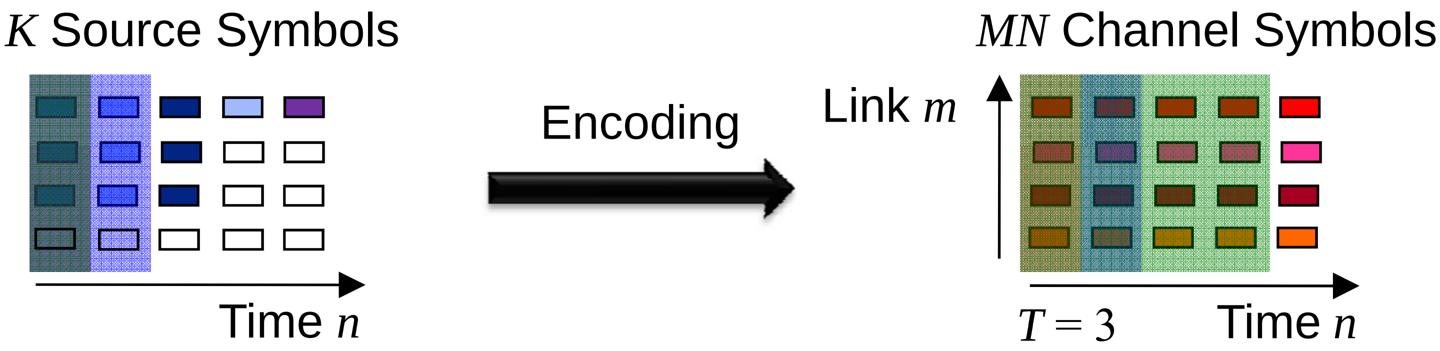


- Considerations
  - Recovery from link outages and bursty packet erasures
  - Minimize decoding delay for each source packet
  - Minimize code redundancy
- What is the best way to introduce redundancy into packets?

Related Work

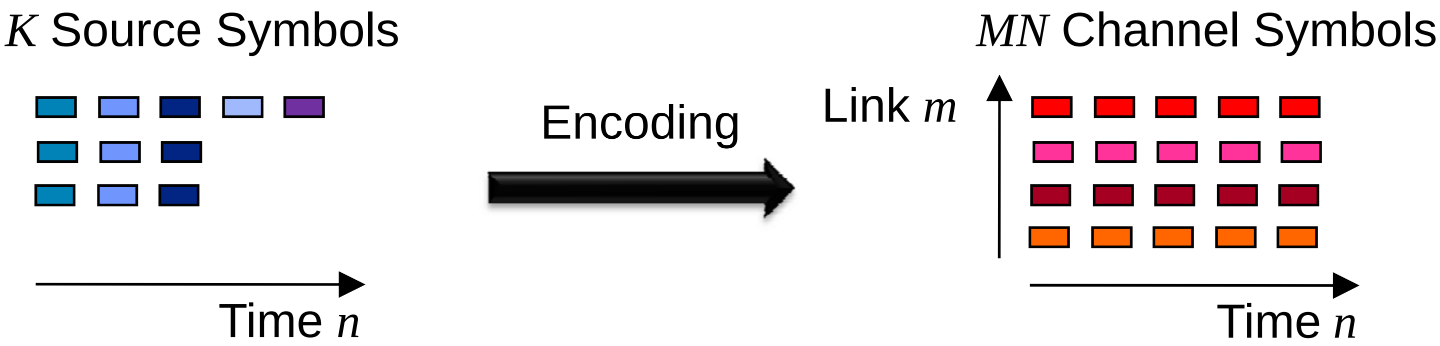
- Delay-optimal streaming codes
  - [Martinian et al. 2002, 2004, 2007]
- Streaming codes for broadcast channel
  - [Badr, Khisti, Martinian 2010]
- Our work is different from previous work
  - Multiple bursts, link outage (vs. single burst)
  - Causal codes (vs. systematic codes)
  - Block codes (vs. streaming codes)

Problem Formulation



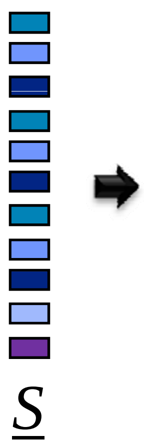
- Symbols over finite field Q
- Links $m = 1, \ldots, M$; time $n = 1, \ldots, N$
- Block codes: K source symbols encoded into MN channel symbols
  - Rate $R := K / N$ symbols/unit time
- Code is *causal*
- Correct up to Z bursts of length B symbols and L link outages
- Each source symbol is decoded with decoding delay T
- Code is *Singleton-achieving* if $MN = K + LN + BZ$ (thus minimum redundancy)

Problem Formulation

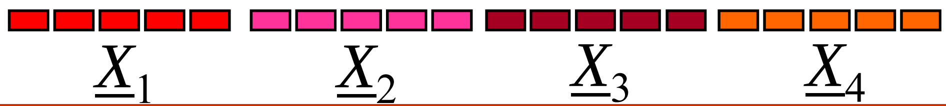


- If code is linear, encoding can be represented in matrix multiplication:

$$\underline{S}\cdot(G_1 \quad G_2 \quad \ldots \quad G_M) = (\underline{X}_1 \quad \underline{X}_2 \quad \ldots \quad \underline{X}_M)$$



$$\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{pmatrix}$$



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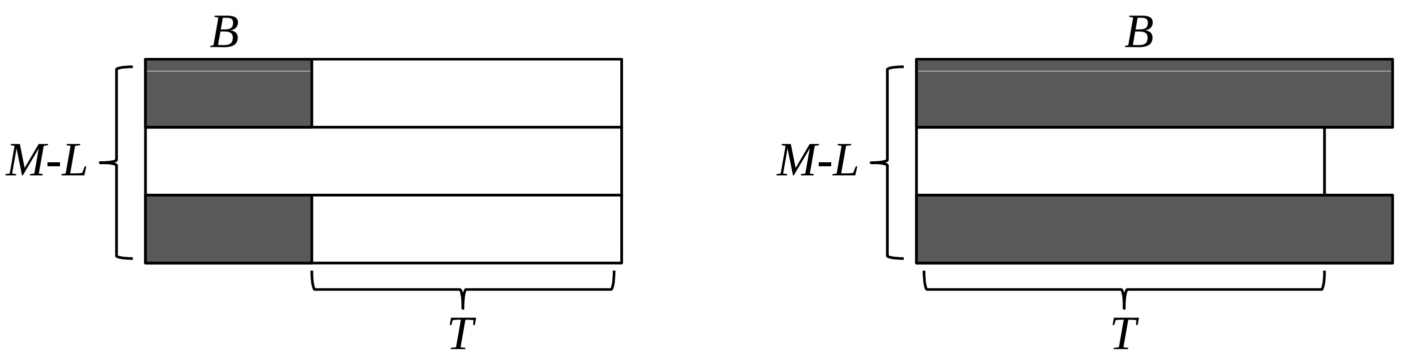
Main Result

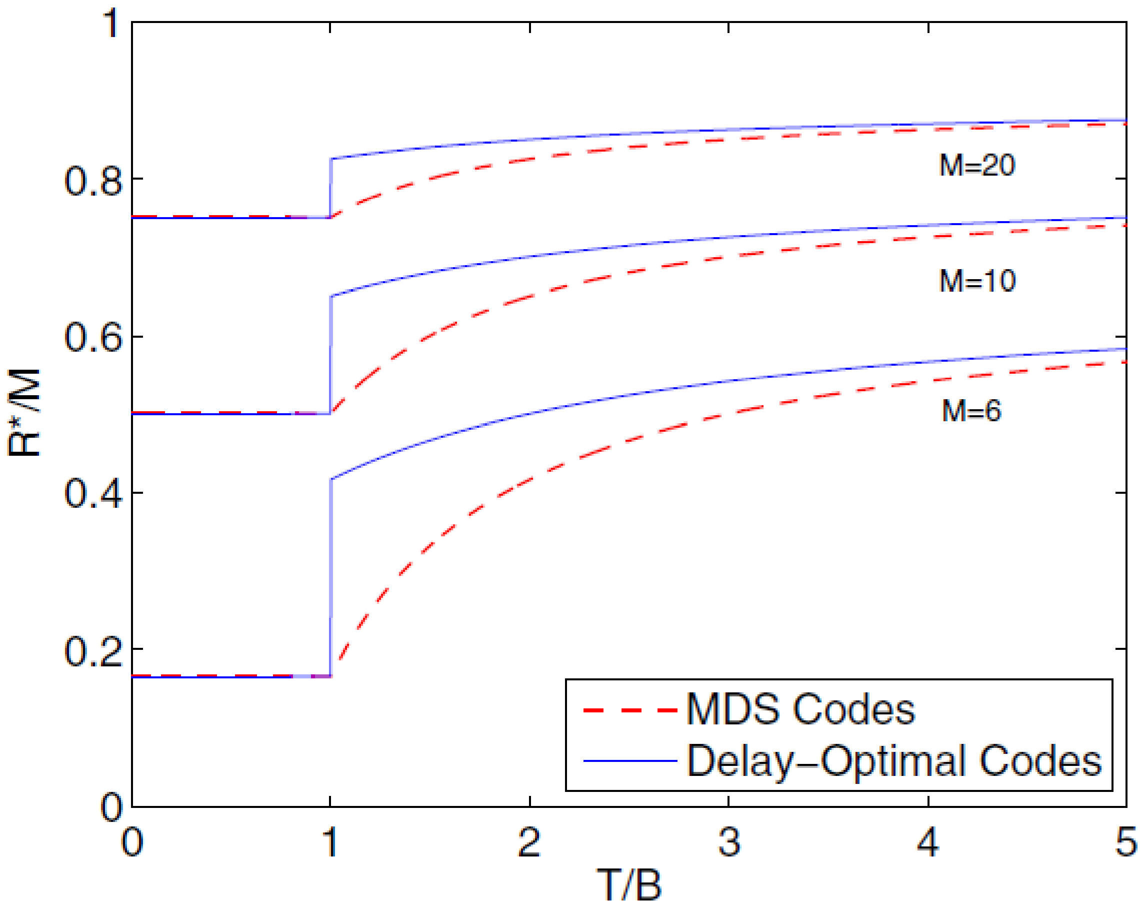
- Theorem (Rate-delay tradeoff)
  - It is possible to construct a delay-T rate-R Singleton-achieving M-parallel-link causal block erasure code feasible for L link outages and Z bursts of length B, each occurring on a separate link, if

$$R \le R^* := \begin{cases} M - L - \dfrac{ZB}{T+B}, & T \ge B \\ M - L - Z, & T < B, \end{cases}$$

  - Conversely, if $R > R^*$, no feasible code can be constructed.



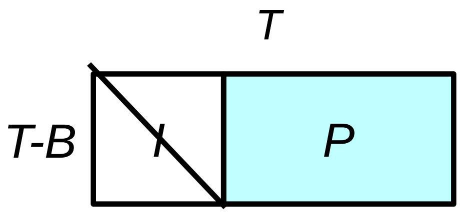
Numerical Bound



- Rate R
- Delay T
- # Links M
- Burst Length B
- # Link Outages $L = 2$
- # Bursts $Z = 3$

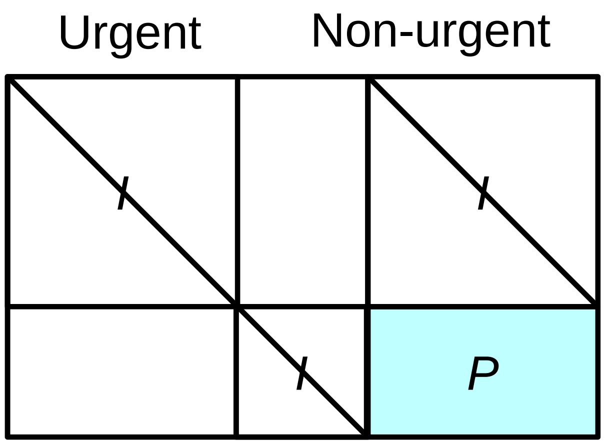
Single-Link Code Construction

[Martinian et al. 2002]



$$\begin{pmatrix} I_{T-B} & P_{(T-B)\times B} \end{pmatrix}$$

(n,k)=(T,T-B) cyclic code



$$G_D = \begin{pmatrix} I_T & \begin{matrix} I_B \\ P_{(T-B)\times B} \end{matrix} \end{pmatrix}$$

(T+B,T) single-link
delay-optimal code

Parallel-Link Code Construction

- Linear parallel-link codes can be constructed based on single-link codes

$$G = \begin{pmatrix} U \otimes G_D \\ V \otimes \begin{pmatrix} 0_{B\times T} & I_B \end{pmatrix} \end{pmatrix} = \begin{pmatrix} G_1 & G_2 & \dots & G_M \end{pmatrix}$$

  - $G_D \sim (T+B, T)$ single-link delay-optimal code (inner code)
  - $U \sim (M, M-L)$ MDS code (outer code)
  - $V \sim (M, M-L-Z)$ MDS code (outer code)
  - $\otimes$: Kronecker product
  - Use G_m for Link m, $m = 1,\dots,M$
- Claims
  - Corrects L link outages and Z B-bursts with delay T
  - Has rate $R = R^*$

Example (2 Bursts, 1 Outage)

$$
\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{pmatrix}
$$

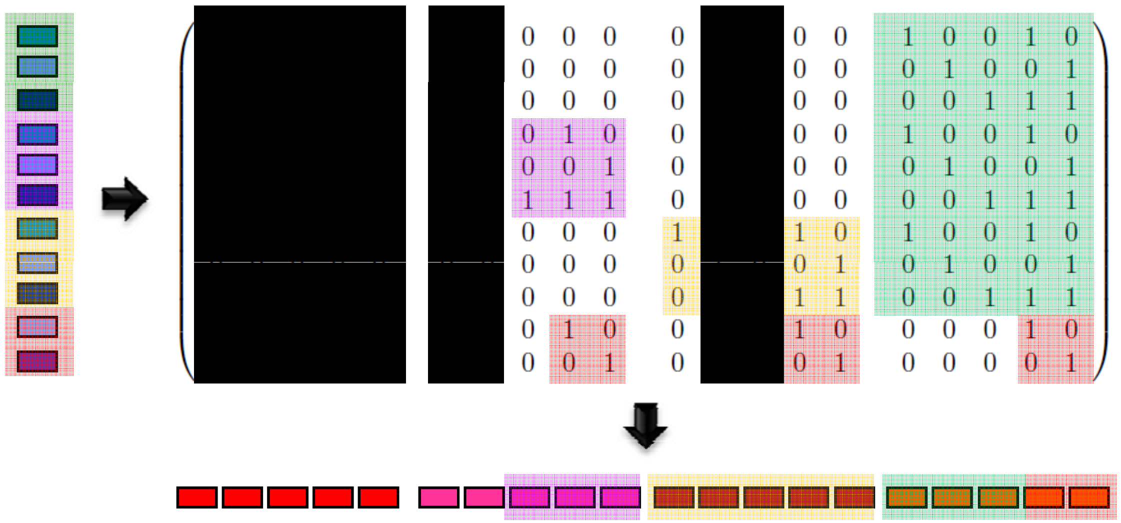
$$
G = \begin{pmatrix} U \otimes G_D \\ V \otimes \left(0_{B\times T} \quad I_B\right) \end{pmatrix} = \left(G_1 \quad G_2 \quad \cdots \quad G_M\right)
$$

$$
G_D = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\qquad
U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}
$$

$$
V = \left(1\ 1\ 1\ 1\right)
$$

- Rate $R=11/5$
- Delay $T=3$
- # Links $M=4$
- Burst Length $B=2$
- # Bursts $Z=2$
- # Link Outages $L=1$

Decoding Example



- Rate $R=11/5$
- Delay $T=3$
- # Links $M=4$
- Burst Length $B=2$
- # Link Outages $L=1$
- # Bursts $Z=2$

The Converse

Singleton-achieving codes with

$$R > R^* := \begin{cases} M - L - \dfrac{ZB}{T+B}, & T \geq B \\ M - L - Z, & T < B, \end{cases}$$

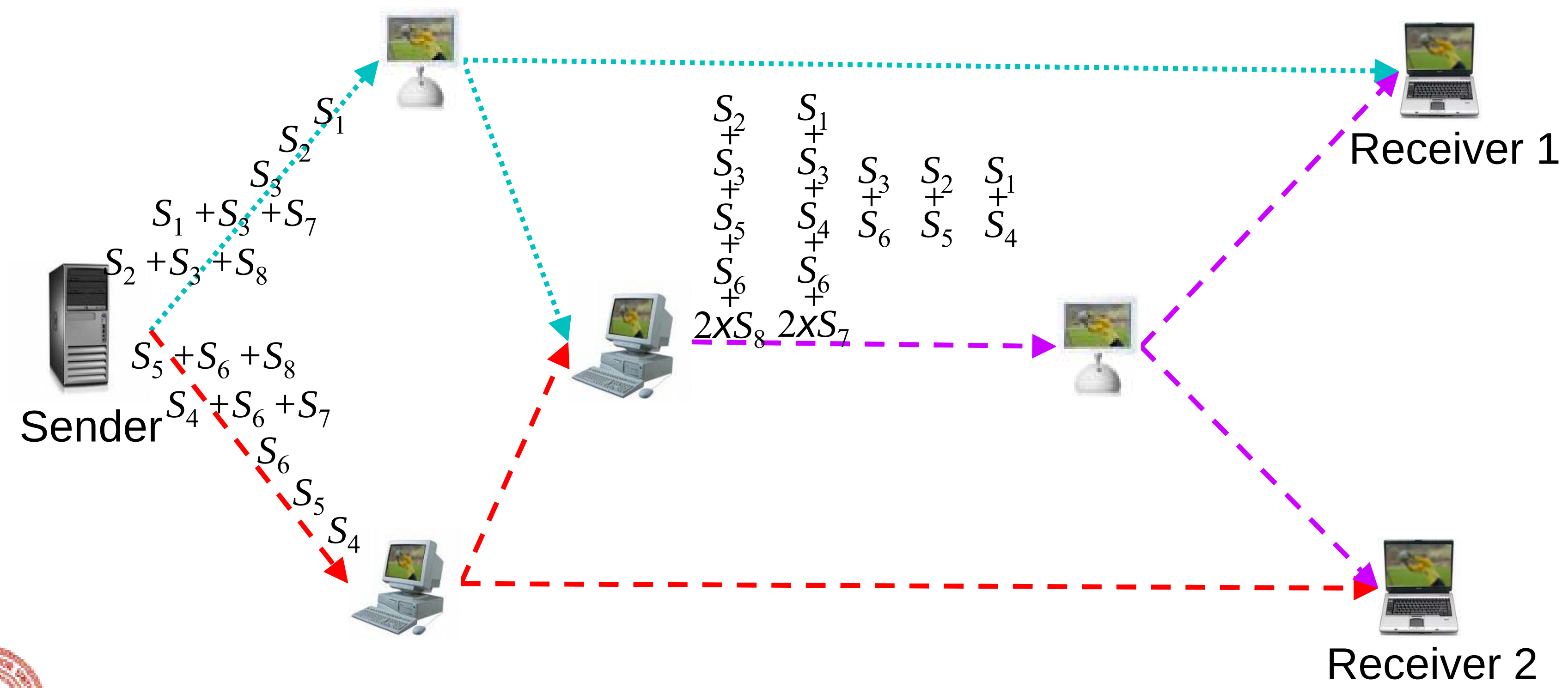
do not exist.

- Proof outline
  - Feasibility of code must be independent of source symbols' prior distribution
  - Assume source i.i.d. uniform over Q
  - Show contradiction via entropy inequalities

- Rate R
- Delay T
- # Links M
- Burst Length B
- # Link Outages L
- # Bursts Z

Extension to Multicast Networks: A Butterfly Network Example

- Multicast 8 source symbols to two receivers
- Decoding delay 3
- Correct up to one burst of length 2 occurring in any one link



Conclusions

- Delay-optimal codes for parallel links are both theoretically interesting and practically relevant
- Many applications
  - Live video streaming with path diversity
  - Peer-assisted packet loss repair for IPTV
  - Ultra-low delay trading systems
  - …
- Future work
  - Generalization to multicast network
  - Consideration of link delay
  - Non-Singleton-achieving case

Thank you !

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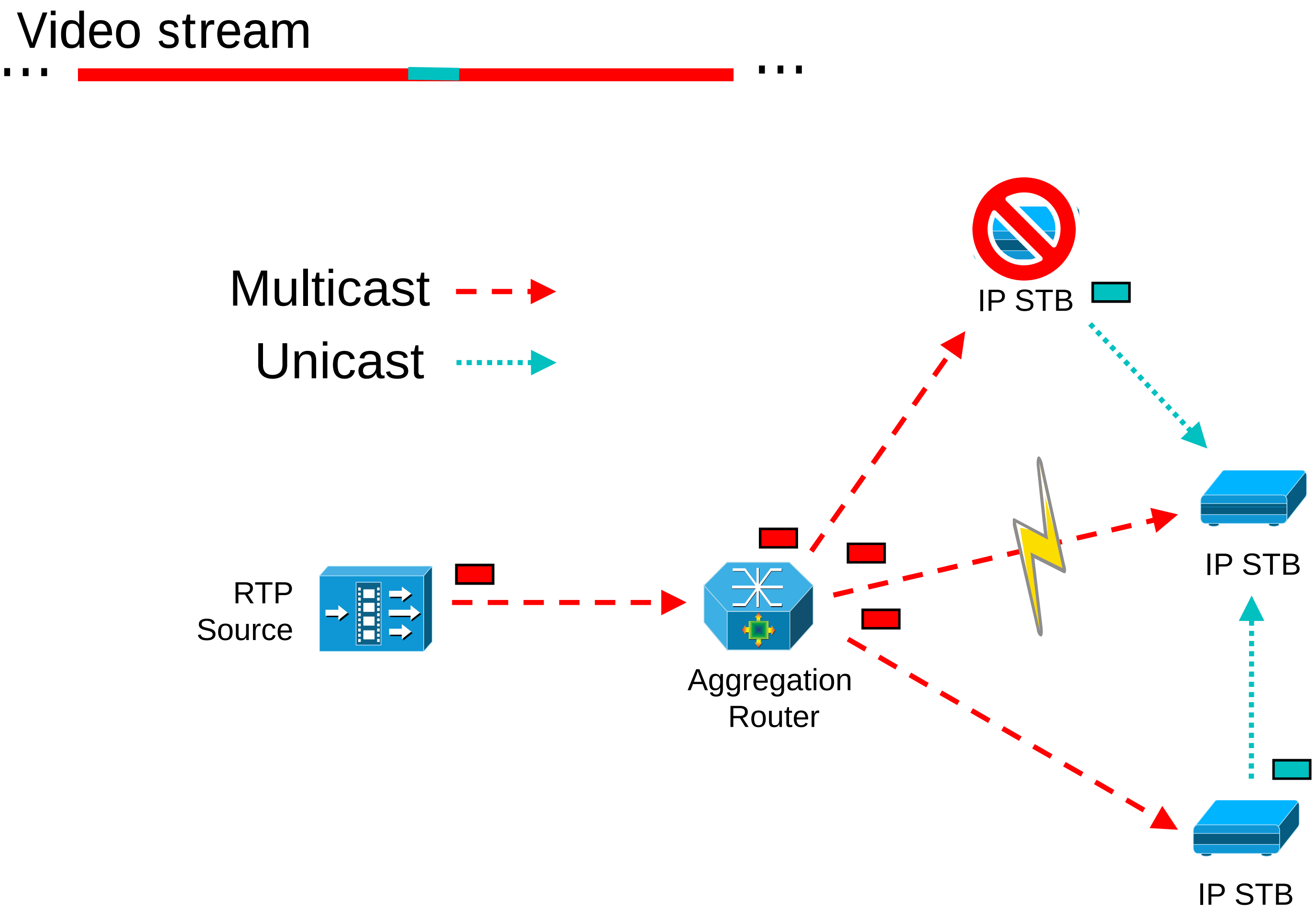
Backup Slides

Peer-Assisted Packet Repair for IPTV



[Li, Zhu, Begen, Girod, 2009]
[Azgin, Altunbasak, 2010]

The Converse

$$R \le R^* := \begin{cases} M - L - \dfrac{ZB}{T+B}, & T \ge B \\ M - L - Z, & T < B, \end{cases}$$

$\Leftarrow$

$$MN = K + LN + BZ$$

(Singleton-achieving)

$$R = \frac{K}{N} \quad \text{(Rate)}$$

$$\begin{cases} T \ge N - B, & T \ge B \\ N \le B, & T < B, \end{cases}$$

- Proof technique
  - Code feasibility must be independent of source prior distribution
  - Assume source i.i.d. uniform over Q
  - Show contradiction via entropy inequalities

- Rate R
- Delay T
- # Links M
- Burst Length B
- # Link Outages L
- # Bursts Z

Case I: $T \geq N\text{-}B, T \geq B$

$S_1 \quad S_2 \quad S_3 \quad S_7 \quad S_8$

$S_4 \quad S_5 \quad S_6$

[redacted]

$X_9 \quad X_{10}$

S_1, S_4

- Example:
  - Block length $N = 5$, # Links $M = 2$,
  - # Link outages $L = 0$, # Bursts $Z = 1$, Burst length $B = 2$
  - Want to show: $T \geq 3$
- Proof by contradiction
  - Assume $T = 2$, early recovery implies:

$$H(S_1, S_4 \mid X_3, X_6^8) = 0$$
$$H(S_1, S_4 \mid X_1^3, X_8) = 0$$
$$H(S_1, S_4 \mid X_1^2, X_6^8) = 0$$
$$H(S_1, S_4 \mid X_1^3, X_6^7) = 0$$

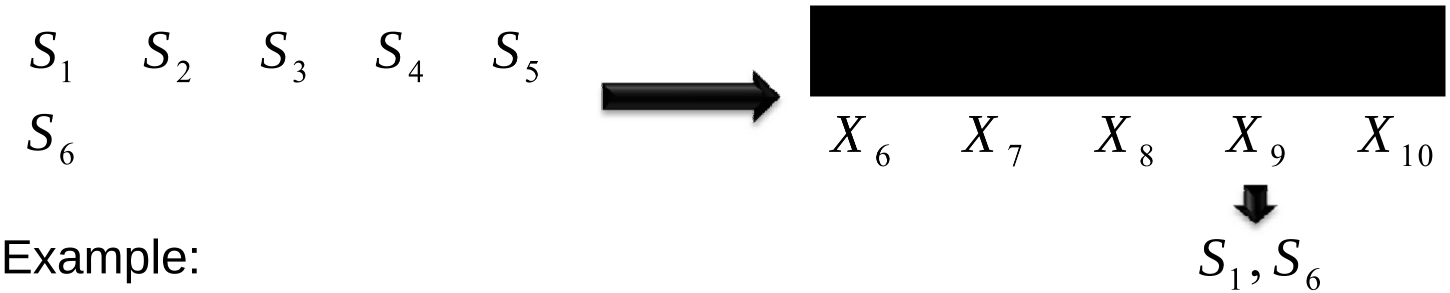
$$\Rightarrow \quad H(X_1^3, X_6^8) < \sum_{i=1,2,3,6,7,8} H(X_i)$$

  - Full recovery implies:

$$H(S_1^8 \mid X_1^3, X_6^{10}) = 0 \quad \Rightarrow \quad H(X_1^3, X_6^{10}) \geq H(S_1^8)$$

Contradiction !!

Case II: $N \leq B, T < B$



- Example:
  - Block length $N = 5$, # Links $M = 2$,
  - # Link outages $L = 0$, # Bursts $Z = 1$, Burst length $B = 4$
  - Want to show: if $T = 3$, contradiction
- Proof by contradiction
  - Assume $T = 3$, early recovery implies:

$$H(S_1, S_6 \mid X_6^9) = 0$$

  - Full recovery implies:

$$H(S_1^6 \mid X_1, X_6, X_7^{10}) = 0$$

  - But:

$$(X_1, X_6) = f(S_1, S_6) = f(g(X_6^9))$$

Contradiction !!