Solving the Single Rate 2-pair Network Coding Problem With the $\mathcal{A}$-set Equation

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Outline of Presentation

- Introduction

Outline of Presentation

- Introduction
- Proof of the Main Result

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- Proof of the Main Result
- Discussions

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1 Introduction

Definition

A single rate 2-pair network coding (S2PNC) problem is specified as follows:

1. A communication network $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$.
2. Two desired unit rate flows from s_i to t_i for $i = 1, 2$.

Remark: we assume each link $e \in E$ has the unit capacity and s_i has no in-edge and a single out-edge and t_i has no out-edge and a single in-edge for $i = 1, 2$.

If the desired flows can be established by network coding, then the S2PNC problem is called solvable, otherwise, it is called unsolvable.

Known Result

A S2PNC problem is solvable if and only if one of the following two (controlled edge overlap) conditions holds.

- [Condition 1] There exists a collection $\mathcal{P}$ of two paths P_{s_1,t_1} and P_{s_2,t_2}, such that $\max_{e \in E} es_{\mathcal{P}}(e) \leq 1$.
- [Condition 2] There exist a collection $\mathcal{P}$ of three paths $\{P_{s_1,t_1}, P_{s_2,t_2}, P_{s_2,t_1}\}$, and a collection $\mathcal{Q}$ of three paths $\{Q_{s_1,t_1}, Q_{s_2,t_2}, Q_{s_1,t_2}\}$, such that $\max_{e \in E} es_{\mathcal{P}}(e) \leq 2$ and $\max_{e \in E} es_{\mathcal{Q}}(e) \leq 2$.

[1] C.-C. Wang and N.B. Shroff, “Beyond the butterfly-a graph theoretic characterization of the feasibility of network coding with two simple unicast sessions,” ISIT 2007.

$\mathcal{A}$-set

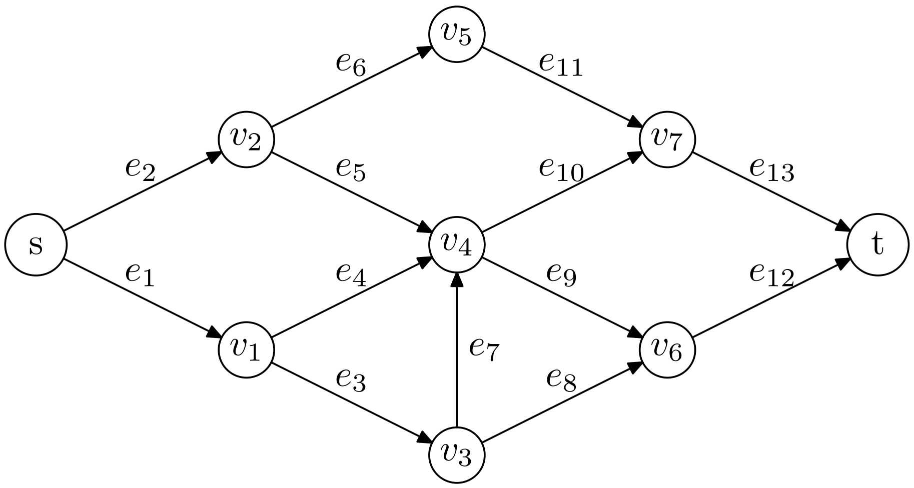
Definition 1.1. *Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network with maximal flow f, then the $\mathcal{A}$-set of $\mathcal{N}$ is defined as*

$$\mathcal{A} = \bigcup_{\langle V_s, V_t \rangle is Min-Cut} \langle V_s, V_t \rangle.$$

- $\mathcal{A}$-set is composed of the most “important” links of the network.

$\mathcal{A}$-set

Example:



- $\mathcal{A} = \{e_1, e_2, e_{12}, e_{13}\}$

The Method

- Decomposed $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ into four point-to-point networks $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$, $i, j = 1, 2$.
- The point-to-point network is easy to deal with.
- The relations of $\mathcal{A}_{i,j}$ can completely determine the solvability of $\mathcal{N}$, where $\mathcal{A}_{i,j}$ is the $\mathcal{A}$-set of $\mathcal{N}_{i,j}$.

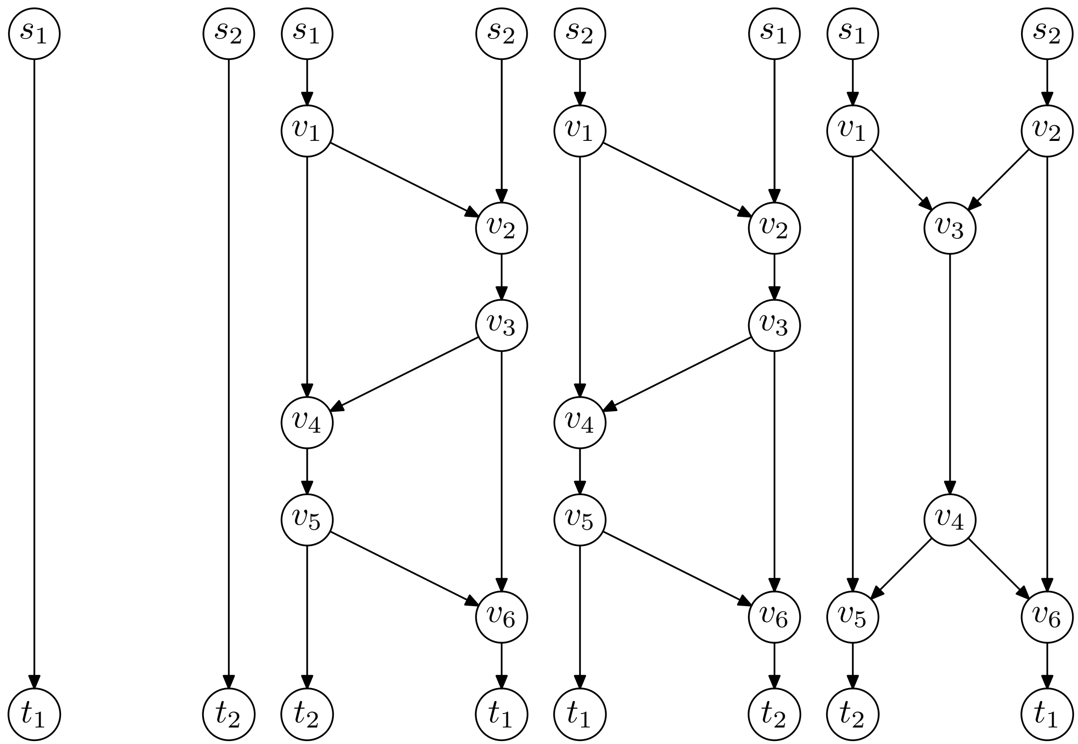
Main Result

- Let $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ be the underlying 2-pair network. The S2PNC problem is solvable if and only if $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$.
- We call $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$ as the $\mathcal{A}$-set equation of $\mathcal{N}$.

2 Proof of the Main Result

Basic Configuration

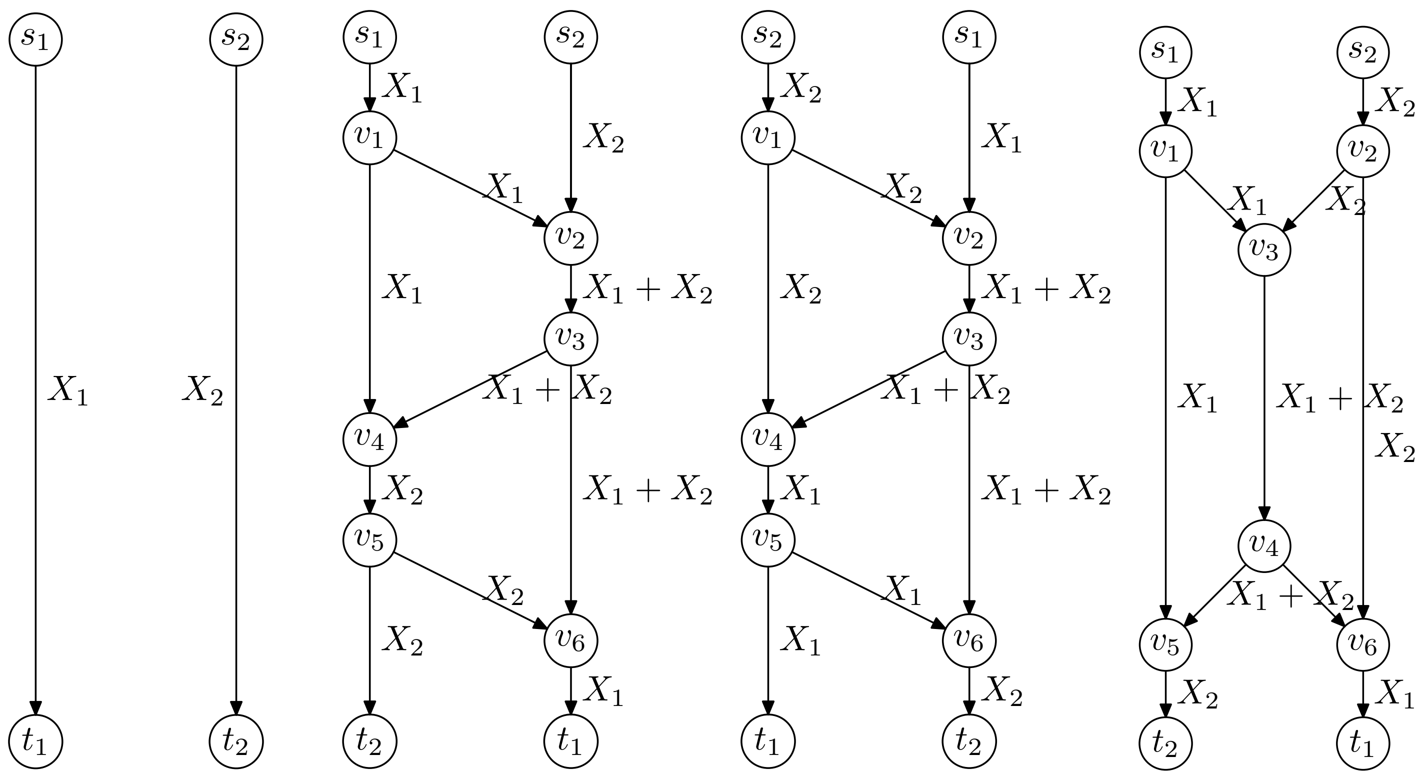
We call (a), (b), (c), (d) as the basic configuration of solvable single rate 2-pair networks.



(a) (b) (c) (d)

Lemma

If $\mathcal{N}$ contains (a), (b), (c), or (d), then $\mathcal{N}$ is solvable.



(a) (b) (c) (d)

What does "Contain" mean?

We say $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ contains $\mathcal{N}_0 = (V', E', \{s'_1, s'_2\}, \{t'_1, t'_2\})$ if there exists a function f from the edges of $\mathcal{N}_0$ to the paths of $\mathcal{N}$ such that:

(1) If $s'_i \longrightarrow s_i$, $t'_i \longrightarrow t_i$, for $i = 1, 2$;

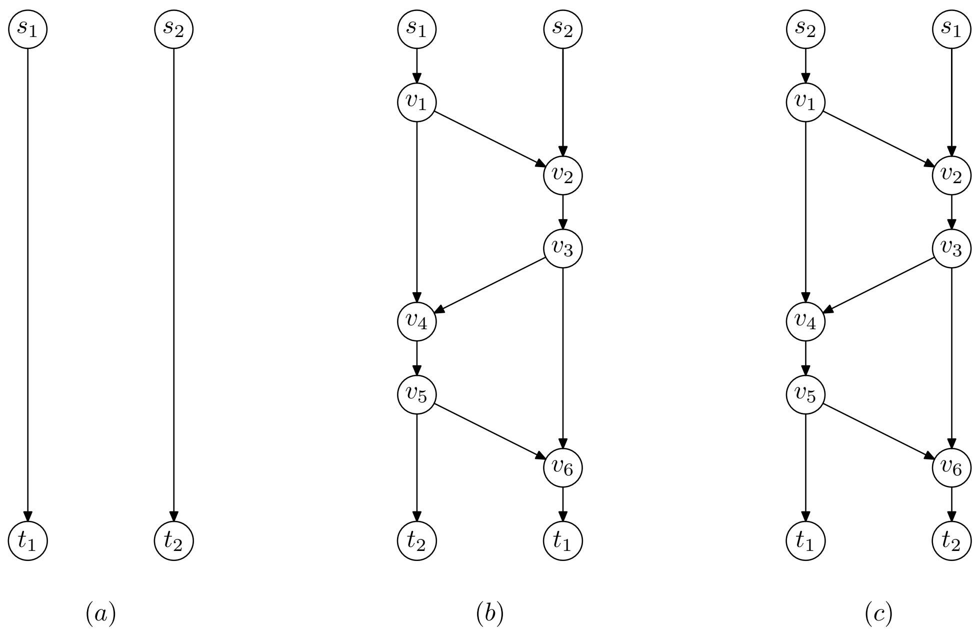
(2) If $head(e'_1) = tail(e'_2)$, then $head(f(e'_1)) = tail(f(e'_2))$, for $e'_1, e'_2 \in E'$;

(3) If $e'_1 \neq e'_2$, then $f(e'_1)$ and $f(e'_2)$ are edge-disjoint, for $e'_1, e'_2 \in E'$.

Proof of "Necessity"

Case 1: $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$

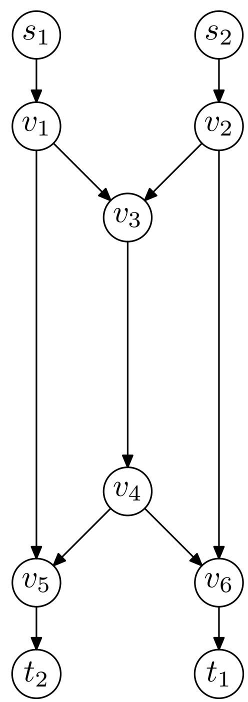
Theorem 2.1. *Let* $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ *be a* 2*-pair unicast network. If* $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$*, then the network contains (a),(b), or (c).*



Proof of “Necessity”

Case 2: $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$

Theorem 2.2. *If* $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$ *and* $(\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) \cap (\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) = \emptyset$. *Then* $\mathcal{N}$ *contains Fig. (d).*



(d)

Proof of “Sufficiency”

◇ Suppose $\mathcal{N}$ is solvable.
Consider two cases: (1) $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$; and (2) $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$.

(1) If $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$, then $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$.

(2) If $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$, then one can find an $s_1 - t_2$ path and an $s_2 - t_1$ path disjoint with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$.

◇ $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$ if and only if
there exist an s_1-t_2 path P_1 and an s_2-t_1 path P_2 such that $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$.

A detailed proof was available at:
http://arxiv.org/abs/1007.0465

3 Discussions

Capacity Factor

Definition:

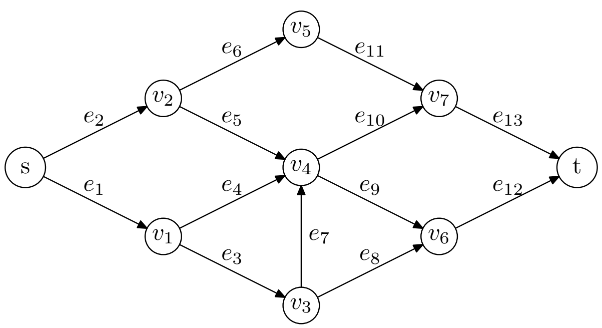
Definition 3.1. *Let* $\mathcal{N} = (V, E, s, t)$ *be a directed acyclic network with node set* V*, link set* E*, source node* s*, sink node* t*.* $F \subseteq E$ *is called a capacity factor (CF) of* $\mathcal{N}$ *if and only if the following two conditions are satisfied.*

1. $C_{\mathcal{N}\backslash F}(s,t) < C_{\mathcal{N}}(s,t)$;
2. $C_{\mathcal{N}\backslash F'}(s,t) = C_{\mathcal{N}}(s,t)$ for any $F' \subsetneq F$,

[2] K. Cai and P. Fan, "An algebraic approach to link failures based on network coding," *IEEE Trans. Inf. Theory,* vol. 53, no. 2, pp. 775-779, Feb. 2007.

Capacity Factor

Example:



- $\{e_1\}$, $\{e_2\}$, $\{e_{12}\}$, $\{e_{13}\}$
- $\{e_3, e_4\}$, $\{e_3, e_9\}$, $\{e_5, e_6\}$, $\{e_5, e_{11}\}$, $\{e_6, e_{10}\}$, $\{e_8, e_9\}$, $\{e_{10}, e_{11}\}$
- $\{e_4, e_7, e_8\}$

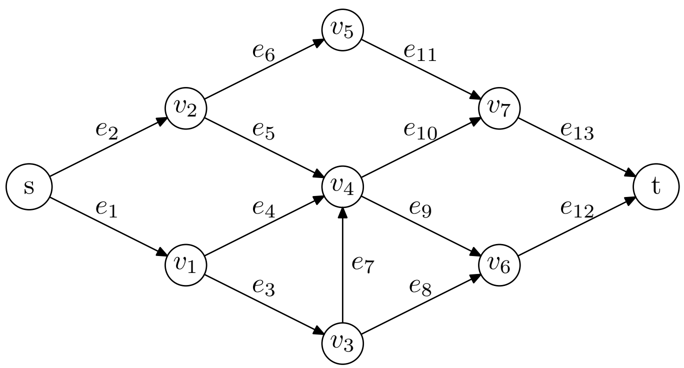
Capacity Rank

Definition:

- Capacity rank: the minimal cardinality of the CFs containing e; denoted by $CR(e)$.
- The capacity rank characterize the *degree of importance* of a link, i.e., the smaller capacity rank a link has, the more important the link is.

Capacity Rank

Example:



- $CR(e_1) = CR(e_2) = CR(e_{12}) = CR(e_{13}) = 1$,
- $CR(e_3) = CR(e_4) = CR(e_5) = CR(e_6) = CR(e_8) = CR(e_9) = CR(e_{10}) = CR(e_{11}) = 2$,
- $CR(e_7) = 3$, with the least importance.

Questions

- The links in the $\mathcal{A}$-set is just the links with capacity rank 1. That is to say, the solvability of the S2PNC problem is determined by the most important links of $\mathcal{N}$.
- Can he solvability of a general multi-source multi-sink network coding problem be determined by the links with capacity ranks $\leq k$, a small integer?
- Determine k or give bounds on k?
- Is there a similar equation for the k-pair case?

Thank You & Questions